Solving Subtraction Problems by Means of Indirect Addition

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Subtraction problems of the type $a - b = ?$ can be flexibly solved by various strategies, including the indirect addition strategy ("how much do I have to add to $b$ to get at $a$?"). Little research has been done on the use of the indirect addition strategy with multi-digit numbers. The present literature review entails a summary of three recent and closely related studies conducted by the authors on this issue. The results of our first study revealed that young adults efficiently and flexibly applied indirect addition on 3-digit subtractions. The results of our second and third study showed that elementary school children seldom used indirect addition on 2-digit subtractions, despite its computational efficiency. This held true even in children who received school-based instruction in the strategy. We end with a discussion of some theoretical, methodological, and educational implications of the studies being reviewed.

THEORETICAL AND EMPIRICAL BACKGROUND

Instructional psychologists and mathematics educators have long emphasized the educational importance of fostering flexibility in children’s self-constructed strategies as a key step in improving (elementary) mathematics education (see, e.g., Brownell, 1945; Freudenthal, 1991; Thompson, 1999; Wittmann & Müller, 1990). Since the turn of the century, many curriculum reform documents, innovative curricula, textbooks, software, and other instructional materials have been based on the belief that promoting strategy flexibility is important for all children, including younger and mathematically weaker children (Baroody, 2003; Kilpatrick, Swafford, & Findell, 2001; Verschaffel, Greer, & De Corte, 2007; Verschaffel, Greer, & Torbeyns, 2006). However, systematic and well-designed research that convincingly supports these basic claims is still rather scarce (Star, 2005).

One of the curricular domains in which variation and flexibility in strategy use have been intensively investigated is multi-digit addition and subtraction (Verschaffel et al., 2006; Verschaffel, Greer, & De Corte 2007). Such problems can be solved by two types of strategies (Beishuizen, 1993, 1999; Blöte, Klein, & Beishuizen, 2000; Blöte, Van der Burg, & Klein, 2000.

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One type is the split method (or decomposition, partitioning, or combining-units-separately method), in which units, tens, hundreds, and so on of both numbers are split off and handled separately (e.g., $86 - 25 = ?$ is determined by taking $80 - 20 = 60$ and $6 - 5 = 1$, answer $60 + 1 = 61$). The second is the jump method (or cumulative, sequential, or begin-with-one-number method), in which the values of each digit of the second number are counted up or down from the first unsplit number (e.g., $86 - 25 = ?$ is determined by taking $86 - 20 = 66$ and $66 - 5 = 61$). In addition to these two types of strategies (and some mixed strategies combining elements of both), some researchers (e.g., Buys, 2001; Fuson et al., 1997; Torbeyns et al., 2006; Treffers, 2001) have identified a third type, called varying or compensation strategies (e.g., $86 - 25 = 85 - 25 + 1 = 60 + 1 = 61$).

All three strategies discussed in the previous paragraph belong to the direct subtraction (DS) class of strategies, in which the subtrahend is subtracted from the minuend (e.g., $71 - 29 = ?$; $71 - 20 = 51$, $51 - 9 = 42$). Another class of strategies involves indirect addition (IA) strategies, in which the result of a subtraction is found by determining how much should be added to the subtrahend to arrive at the minuend (e.g., $71 - 59 = ?$; $59 + 10 = 69$, $69 + 2 = 71$, so the answer is $10 + 2$ or $12$).\footnote{It should be noted that there is also a third class of strategy, namely the indirect subtraction strategy, in which one finds the solution by determining how much has to be decreased from the larger given number to get the smaller one ($81 - 72 = ?$; $81 - ? = 72$) (De Corte & Verschaffel, 1987).} Beishuizen (1997) referred to such strategies as solving subtractions by means of addition, Blöte et al. (2000, 2001) used the term short jump strategy, Brissiaud (1994) called it working forward (as opposed to working backward, the term he uses for the direct subtraction strategy), Menne (2001) and Veltman and Treffers (1995) used the term working from the beginning (versus working from the end), Selter (2001) called it the adding-up or completion strategy, whereas Van den Heuvel (personal communication, 03-01-2008) suggested the term adding-on or filling-up strategy. In the remainder of this article we will use the term indirect addition strategy.

Until now, research on children’s strategies for doing multi-digit subtractions has concentrated on rational and didactical analyses of the relative strengths and weaknesses of the split and jump strategy, on the relative frequency and efficiency (in terms of accuracy and/or speed) with which children use these two types of strategies, on how particular task and subject variables affect the efficiency and/or selection of both types of strategies, and on the impact of certain features of traditional or experimental instructional settings on these strategy performance characteristics (Beishuizen, 1993, 1999; Blöte et al., 2000, 2001; Buys, 2001; Fuson, 1992; Fuson et al., 1997; Klein et al., 1998; Selter, 1998; Torbeyns et al., 2006; Treffers, 2001).

Relatively little attention has been paid to the distinction between DS and IA strategies. This is remarkable for two reasons. One is that IA seems to have some computational advantages over DS, at least for a particular kind of subtraction problem, namely those with a relatively small difference between the subtrahend and minuend, such as $81 - 79 = ?$ or $452 - 444 = ?$ (Beishuizen, 1997; De Corte & Verschaffel, 1987; Menne, 2001; Thornton, 1990; Treffers, 2001). Indeed, a rational analysis shows that subtractions with a small difference between the minuend and the subtrahend allow the child to determine this difference with a relatively quick and easy counting or adding on process (e.g., $81 - 79 = ?$; $79 + 1 + 1 = 81$ so the answer is $2$),
while counting down or subtracting the subtrahend from the minuend would take considerably more and/or more difficult steps (e.g., $81 - 79 = ?$; $81 - 70 = 11$ and $11 - 9 = 2$). Although didactical publications provide an (intuitively convincing) demonstration of the computational efficiency of the IA strategy for extreme examples like $81 - 79 = ?$, they do not contain a systematic attempt to define and justify what exactly is meant by a small difference. Stated differently, these studies do not determine the point from which IA becomes computationally more efficient than DS.\footnote{Such an attempt is made by Groen and Parkman (1972), but only for solving one-digit subtractions like $9 - 2 = ?$ and $9 - 7 = ?$ either through counting down from the larger given number or through counting up from the smaller one.} A second reason is that IA does not only seem to be a computationally efficient strategy but also a very promising strategy from a broader educational perspective. More specifically, the acquaintance with IA as a complementary strategy for DS, the identification of those arithmetic tasks for which it is particularly rewarding, and the reflection on the missing-addend principle that underlie the appropriateness of IA for solving subtraction problems are all valuable activities in the reform-based international strive for developing adaptive expertise in (elementary) mathematics education (Baroody, 2003; Hatano, 2003; Star, 2005; Verschaffel, Torbeyns, De Smedt, Luwel, & Van Dooren, 2007).

The scarce research literature on IA indicates that it is seldom used among traditionally schooled children who did not receive explicit and systematic instruction in this strategy (Klein et al., 1998; Selter, 2001, 2002). Klein and colleagues (1998), for instance, observed that Dutch second graders without explicit and systematic instruction in IA did not solve small-difference subtractions as $71 - 69 = ?$ or $62 - 58 = ?$ with this strategy. The limited number of intervention studies on IA (Blöte et al., 2000; Fuson & Fuson, 1992; Fuson & Willis, 1988; Klein et al., 1998; Menne, 2001; Thornton, 1984, 1990; Veltman & Treffers, 1995) indicate that after explicit instruction in and intensive practice of IA, first and second graders are able to apply this strategy on small-difference subtractions. For example, the intervention study of Klein et al. (1998) revealed that Dutch second graders who had been taught IA on small-difference subtractions in the number domain 20–100 used this strategy on more than 80% of this type of subtractions. Furthermore, after explicit instruction in IA, children’s subtraction performances increased to a level comparable with the (typically better) addition performances, suggesting that IA is indeed a very efficient strategy for doing subtractions (Fuson & Willis, 1988; Klein et al., 1998).

Previous research regarding IA is limited in some ways. First, none of the available studies has made a systematic attempt to analyze the strategy performance characteristics of IA (as compared with DS) on a variety of symbolically presented subtraction items (e.g., on a variety of subtraction items that differ systematically in terms of the size of the difference between minuend and subtrahend). Second, the methodology applied in the available studies that document the rare use of IA does not allow differentiating between various possible reasons for this rarity (e.g., are pupils not applying IA because they do not master it or because they prefer another strategy from their strategy repertoire). Finally, all results that show that the efficient and flexible use of IA can be effectively taught have been collected in developmental studies or teaching experiments in which the instruction was actually provided—or at least directly controlled—by the researchers. It therefore remains unclear to what extent the obtained positive outcomes can be replicated in more regular educational settings.
Taking the above state-of-the-art as our starting point, we set up a research program that aims at more systematically analyzing the (development of the) flexible use of IA on symbolically presented multi-digit subtractions. In one study (Torbeyns, Ghesquière, & Verschaffel, in press), we focused on the (assumed) computational efficiency and flexibility characteristics of IA compared with DS. More specifically, we analyzed the frequency, efficiency, and flexibility with which young adults executed IA and DS strategies on different types of subtractions, using the choice/no-choice method (Siegler & Lemaire, 1997). Based on the favorable results for the IA strategy obtained in this first study, we then conducted two closely related studies on the developmental changes in children’s use of IA, taking into account both age and mathematical achievement level. Whereas the second study involved only traditionally schooled children who had not received any instruction in IA (Torbeyns et al., in press b), the third one compared traditionally schooled children’s use of IA with that of children who worked with a textbook series that paid ample attention to IA (Torbeyns et al., in press a). For technical details about the design and results of the three investigations, please see the original research reports. We end with discussing some theoretical, methodological, and educational implications of the studies being reviewed.

STUDY 1: FLEXIBLE USE OF IA IN YOUNG ADULTS

The aim of our first study (Torbeyns et al., in press) was to analyze young adults’ mastery and use of IA on different types of three-digit subtractions using the choice/no-choice method (Siegler & Lemaire, 1997). Twenty-five university students (from the faculties of Human, Biomedical and Positive Sciences; mostly women; aged from 18 to 26 years) participated in the study. All participants solved a series of 12 three-digit subtractions that consisted of a minuend ranging from 812 to 829. All subtractions required two carries, namely a carry from the tens to the ones and another carry from the hundreds to the tens. We made a distinction among three types of subtractions on the basis of the difference between the minuend and the subtrahend, with four items per subtraction type: (a) subtractions with a small difference between the integers with a subtrahend ranging from 770 to 790 (e.g., 812 – 783 = ?); (b) subtractions with a medium difference between the integers with a subtrahend ranging from 470 to 490 (e.g., 821 – 475 = ?); and (c) subtractions with a large difference between the integers with a subtrahend ranging from 170 to 190 (e.g., 813 – 176 = ?).

All participants solved the series of subtractions individually in one choice and two no-choice conditions. In the choice condition, participants were instructed to solve the 12 subtractions using either DS (i.e., calculating the outcome by subtracting the subtrahend from the minuend; e.g., 812 – 475 = 812 – 400 – 70 – 5 = 337) or IA (i.e., determining how much should be added to the subtrahend to arrive at the minuend; e.g., 812 – 475 = 475 + 25 + 300 + 12 . . . so the answer is 25 + 300 + 12 or 337). In the first no-choice condition participants were instructed to solve all subtractions by means of DS (= DS no-choice condition). In the second no-choice condition they were instructed to answer all subtractions by means of IA (= IA no-choice condition). In all conditions, they had to verbally report the strategy used after each trial. The experimenter registered the accuracy and speed of responding in each condition on a trial-by-trial basis.
Study 1 had four main findings. (a) About two thirds of the participants applied the IA strategy at least once to solve the series of 12 multi-digit subtractions in the choice condition: 48% used alternately DS and IA, whereas 20% answered all items with IA and 32% always applied DS. (b) The frequency of IA and of DS in the choice condition did not differ significantly; participants solved 43% of all items with IA versus 57% with DS. (c) Most importantly, in the two no-choice conditions participants performed equally accurately on both strategies but they were significantly faster with IA than with DS. The nonsignificant interaction between condition and subtraction type indicated that IA was not executed faster than DS only when there was a small difference between the two numbers (e.g., 812 – 783 = ?) but also on the two other subtraction types where the advantage of solving the problem by IA appears to be less straightforward (see Table 1). (d) Participants fitted their strategy choices flexibly to both item and individual strategy performance characteristics: in the choice condition, small-difference subtractions were answered significantly more frequently with IA than the two other problem types, and participants who mastered IA better—as indicated by their strategy speed performances in the no-choice conditions—used this strategy also significantly more frequently in the choice condition.

The results of this first study were replicated in a follow-up study (see also Torbeyns et al., in press) with a similar group of students, similar items (except that this time only subtractions with small to medium differences between the numbers were presented, divided in four problem types on the basis of the size of the difference), and an equal choice/no-choice procedure. The results of this follow-up study were similar to those from the first study, except that in the no-choice conditions the IA strategy was not only executed significantly faster (8.80 s) but also significantly more accurately (87% correct) than the DS strategy (respectively, 16.18 s and 79% correct). Again, there was no significant interaction between condition and subtraction type, neither for accuracy nor for speed, indicating that IA was executed more efficiently than DS not problem types.

### STUDY 2: DEVELOPMENT OF IA IN TRADITIONALLY SCHOoled CHILDREN

The aim of the second study (Torbeyns et al., in press b) was to analyze the developmental changes in the strategy competencies of elementary school children of different ages and different mathematical achievement levels in the domain of addition and subtraction up to 100, with special

<table>
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<tr>
<th>Difference Between Numbers</th>
<th>Direct Subtraction</th>
<th>Indirect Addition</th>
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<tbody>
<tr>
<td></td>
<td>Accuracy</td>
<td>Speed</td>
</tr>
<tr>
<td>Small</td>
<td>83%</td>
<td>12.40 s</td>
</tr>
<tr>
<td>Medium</td>
<td>77%</td>
<td>14.72 s</td>
</tr>
<tr>
<td>Large</td>
<td>85%</td>
<td>13.31 s</td>
</tr>
<tr>
<td>Total</td>
<td>82%</td>
<td>13.47 s</td>
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attention to their application of the IA strategy. The sample of this study consisted of 71 second, 71 third, and 53 fourth graders. At the time of testing, all children had received instruction in two-digit addition and subtraction. A textbook analysis and an interview with the classroom teachers revealed that all children had received instruction that aimed at the fluent application of the jump version of the DS strategy (e.g., solving 45 – 27 = ? by doing 45 – 20 = 25 and 25 – 7 = 18) on all types of two-digit subtractions. Children had not been provided any instruction in other types of strategies to solve such sums (i.e., no instruction of the split, or varying version of DS and no instruction of IA). However, all teachers reported that they allowed their high achievers to apply other strategies than the jump version of the DS strategy on two-digit subtractions, provided that these children had demonstrated (and continued to demonstrate) good mastery of it. Third and fourth graders were also taught the standard paper-and-pencil algorithm for solving multi-digit additions and subtractions. During the introduction phase of these algorithms, these children had been briefly confronted with the split version of the DS strategy as a stepping stone to the subtraction algorithm.

All children individually completed two tasks designed to evaluate their strategy competencies in the domain of addition and subtraction up to 100. The Spontaneous Strategy Use Task consisted of different types of two-digit additions and subtractions, including two-digit subtractions with a small difference between the minuend and the subtrahend (e.g., 41 – 39 = ?), which can be efficiently solved with IA. Children were instructed to solve each item as accurately and as fast as possible with their preferred strategy and to verbally report both the answer and the strategy used immediately after solving each item. The experimenter registered the child’s answer, the speed of responding, and the reported strategy for each child and for each item. The Variability on Demand Task also consisted of various types of two-digit problems, including subtractions with small differences (e.g., 41 – 39 = ?). Children were instructed to solve each item with at least two different strategies and to verbally report each strategy immediately afterward. The experimenter kept asking for another possible strategy until the child had reported the IA strategy, stated that he or she did not know any other strategy, or reported five other alternative solution methods.

Study 2 had three main findings. (a) The analysis of children’s strategy repertoire in the Spontaneous Strategy Use Task revealed that less than 10% of the second and third graders, and only 15% of the fourth graders spontaneously applied the IA strategy at least once to answer the five small-difference subtractions. Thus, children hardly used the IA strategy, even on items in which this strategy was considered to be the most efficient one. (b) All children reported various strategies for solving the two small-difference items from the Variability on Demand Task, but only a minority of them reported IA as an alternative strategy, suggesting that IA was no part of the strategy repertoire of most children. Actually, only 5% of the second graders, 15% of the third graders, and 20% of the fourth graders mentioned the IA strategy at least once as a possible alternative solution strategy. (c) Despite the low overall frequency of the IA strategy in the two tasks and in the three age groups, there tended to be an impact of mathematical achievement.

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3This study focused on the development of two so-called clever strategies, namely IA and the compensation strategy (e.g., solving 45 – 19 = ? by doing 45 – 20 + 1 = 26). Taking into account the scope of the present article, we only refer to the part about the IA strategy.

4Because of the very low frequency of children’s use of the IA strategy, it was not possible to perform a reliable statistical analysis of the flexibility with which children of different ages and different achievement levels applied or suggested IA. Consequently, all findings are based on a descriptive quantitative analysis of children’s strategy use.
level on the frequency with which IA was spontaneously used (in the Spontaneous Strategy Use Task) or mentioned as an alternative solution method (in the Variability on Demand Task): higher achievers tended to solve more small-difference items with the IA strategy than lower achievers.

Why did we find so little evidence for the (flexible) use of the IA strategy in our sample? The strong instructional focus on the routine mastery of the DS strategy for all types of two-digit subtractions—following an instructional approach that blends elements of a skills and a conceptual approach to mathematics instruction, to use Baroody and Coslick’s (1998) terminology—might account for this result. By focusing on the mastery of this single standard strategy to solve all types of additions and subtractions up to 100, teachers—indirectly, but perhaps sometimes even directly—communicated that the use of one single strategy on all sums was valued most, affecting children’s beliefs about how to solve such sums as well as their actual learning and problem-solving processes (see also Baroody & Coslick, 1998; Schoenfeld, 1992; Verschaffel, Torbeyns et al., 2007). Moreover, after they arrived in grade 3 and started the preparatory work for the written algorithms, they were confronted with new solution methods (namely, the split version of the DS strategy, followed by the standard subtraction algorithm) that were again certainly not supportive to the emergence of the IA strategy. So, from third grade on, there was an additional instructional factor that did not favor, but rather further hindered the discovery and (efficient and flexible) use of the IA strategy.

STUDY 3: DEVELOPMENT OF IA IN VARIOUS INSTRUCTIONAL SETTINGS

In our third study (Torbeyns et al., in press a), we compared the strategy performance on a paper-and-pencil test that comprised four types of subtraction items of second to fourth graders from two Flemish schools, which did not provide any instruction in the IA strategy (= DS-oriented schools), with that of comparable children from a third school in which IA did receive special instructional attention (= IA-oriented school). The difference in instructional approach between the DS- and IA-oriented schools was not only revealed by an analysis of the textbook series used, it was also confirmed by interviews with the teachers, in which they were questioned about various aspects of their instructional approach to (mental) addition and subtraction. According to the textbook analysis and the teacher interviews, first graders from the IA-oriented school were taught to use IA or, as they called it, climbing-up the ladder (instead of descending the ladder, which was used to refer to the DS strategy) when the difference between the two given numbers was small. The textbook also introduced a specific notation for the IA strategy: a little arrow or arc from the subtrahend to the minuend. In second grade, children from this school were taught to solve most two-digit subtraction problems with (the jump version of) the DS strategy, but to switch to IA when the difference between the integers was small (which was operationalised in the textbook as “a difference smaller than 10”). For more details about the instructional approach, see Torbeyns et al., submitted.

In total, 54 second graders, 54 third graders, and 49 fourth graders participated in Study 3. There was a more or less equal number of boys and girls. The number of children from the IA-oriented school and the two DS-oriented schools was, respectively, 53 and 104. Based on their score on a standardized general mathematical achievement test all children were categorized as high, average, or low achievers. At the time of testing, the achievement level for mathematics in the IA-oriented school did not differ significantly from that in the two DS-oriented schools.
All children completed a paper-and-pencil test with 16 two-digit subtraction problems. All integers and results of these 16 items were in the number domain 20–100 and they all involved a carry. We distinguished four types of items on the basis of (a) the difference between the minuend and subtrahend, and (b) the size of the units of the minuend. With respect to the first characteristic, half the items had a difference smaller than 10 (e.g., 81 – 79 = ?), while the other half had a difference between 10 and 20 (e.g., 72 – 58 = ?). Concerning the second characteristic, in half the items the unit of the minuend was small (e.g., 61 – 48 = ?), whereas in the other half it was large (e.g., 95 – 79 = ?). Combining these two task features, we created four item types (n = 4 per type), which are summarized in Table 2.

The test was administered in the children’s regular classroom during a regular mathematics lesson. Children were instructed to solve the problems in whatever way they wanted and to write down their solution strategy in the scrap paper area below each problem. They were told that they were free to report their strategy in the form they found most convenient (e.g., in words, as a number sentence, in schematic form, or using a mixture of forms).

The major result of this study was surprising and, from an instructional perspective, quite disappointing. While children from the IA-favoring school used IA slightly more frequently than children from the two other schools, the frequency of IA was generally extremely low in all schools: 7.53% (or 62 out of a total of 823) for the IA-oriented school and 0.19% (3 out of a total of 1547) for the DS-oriented schools. The results for the impact of the subject and task variables can be summarized as follows. (a) There tended to be some impact of age on the use of IA in the IA-oriented school, since all 62 IA strategies were generated by second graders. The finding that the use of IA tended to drop with grade was in line with our expectation, because in the IA-oriented school the focus of the instructional attention to IA was in grade 2 and decreased strongly in the next grades; but, its complete disappearance after grade 2 was another surprising result. (b) There also seemed to be an impact of achievement level: none of the IA strategy reports were produced by low achieving pupils, whereas average and high achievers solved respectively 10 (1.8%) and 55 (4.75%) of their problems with IA. (c) The task features also seemed to have some influence in the sense that most IA strategies in the school with IA-favoring instruction occurred on the two item types with the smallest difference between minuend and subtrahend (see Table 2).

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Table 2

<table>
<thead>
<tr>
<th>Small Difference</th>
<th>Large Difference</th>
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<tbody>
<tr>
<td>Minuend ending in 1 or 2</td>
<td>Item type 1</td>
</tr>
<tr>
<td>e.g., 81 – 79 = ?</td>
<td>e.g., 72 – 58 = ?</td>
</tr>
<tr>
<td>Minuend not ending in 1 or 2</td>
<td>Item type 3</td>
</tr>
<tr>
<td>e.g., 66 – 58 = ?</td>
<td>e.g., 95 – 79 = ?</td>
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As in the second study, the extremely low overall frequency of IA jeopardized the possibility of performing a reliable statistical analysis of the impact of the instructional, task, and subject factors on the frequency and accuracy of IA. Consequently, we restrict ourselves to a descriptive analysis of the results.
In retrospect, the unexpectedly low frequency of IA strategies in the IA-oriented school was probably due to the features of the instruction in IA as provided by the textbook and as implemented by the teachers. A post-hoc analysis of this instruction suggested that although the IA strategy had been systematically taught and practiced in this school it did not happen in a way that seems necessary to arrive at the meaningful, efficient, and flexible use of the IA strategy. Indeed, the instruction aimed at teaching children how to execute (the jump version of) the IA strategy and at providing them a rule for deciding when to use this strategy that was purely based on the size of the difference between the two given numbers (i.e., “apply IA when the difference between the numbers is less than 10”). Arguably, such a kind of instructional approach in which there is one best strategy for each (type of) sum and in which children receive direct instruction and training in how to select and execute the best strategy for each type of sum (without tolerance for variation in strategy choices and open discussion about this variety), will hardly result in the development of adaptive expertise in (elementary) mathematics (Baroody, 2003; Baroody, Cibulskis, Lai, & Li, 2004; Verschaffel, Torbeyns et al., 2007).

Because we could not exclude that the unexpectedly low number of IA strategies in our third study was due to the technique being used to identify the children’s solution strategies, namely a paper-and-pencil test, we set up a follow-up study with the same children and the same item set, but this time strategy performance was assessed during an individual interview. Unfortunately, only the pupils from the schools without IA instruction could participate in this follow-up study, which took place a few weeks after the first one. The overall frequency of reported IA increased only marginally—from 0.19% in the initial study to 2.43% in the follow-up study—implying that the type of data-gathering method used was not a major cause of the remarkably low frequency of IA observed in the initial study.

CONCLUSION AND DISCUSSION

The IA strategy, particularly with multi-digit numbers, has received little attention from researchers so far. This limited research interest for the IA strategy as a complementary strategy for the conventional DS strategy is quite surprising, especially because there are indications that IA is not only computationally remarkably efficient but also very promising from a broader educational perspective. To enhance our understanding of the characteristics and use of the IA strategy, we conducted three closely related studies on the developmental changes in the efficiency and flexibility with which elementary school children and young adults apply this strategy, which we have reviewed in the present article.

Participants in the first study (Torbeyns et al., in press) may have performed significantly better when asked to execute the IA strategy than when applying the DS strategy, even on problems where the advantage of solving the sum by IA seems unclear (e.g., 782 − 257 = ?), because working upwards continues to be cognitively easier than working downwards, even among arithmetically experienced young adults. Furthermore, this study revealed that young adults flexibly fit their strategy choices to individual strategy performance characteristics, meaning that the IA strategy was used most frequently by those individuals who profited most from its use (as compared with using DS).

Whereas our first study provided evidence for computational efficiency and flexible application of the IA strategy among young adults, our second and third study revealed that second-, third-, and
fourth-grade children used the IA strategy rarely. Moreover, confronting these children with problems with a small difference between minuend and subtrahend, inviting them to generate other strategies than the one spontaneously applied, and even working with a mathematics textbook series that pays explicit and systematic attention to IA had only a limited positive effect on the frequency of IA strategy use. Why did the children from Study 2 and 3 apply the IA strategy so seldom? A first possible explanation relates to the children’s limited general cognitive resources—whether these limitations are conceived in traditional Piagetian terms (e.g., the lack of reversibility in their logico-mathematical thinking; Piaget, 1965), in terms of their general processing potentials (such as speed of processing, attention, and working memory; Demetriou, 2004), or in terms of their metacognitive capacities (Flavell, 1982)—or to their more restricted specific attentional resources due to the fact that they were still struggling with the mastery of the DS strategy, as Siegler’s (1998) Strategy Choice and Discovery Simulation (SCADS) model of cognitive strategy change would predict. According to the latter model, the more frequently one executes a particular strategy (e.g., the jump version of the DS strategy) the more strongly the operative parts of that strategy become associated and, thus, the less attentional resources one needs to monitor this strategy’s execution. This allows the individual to allocate these freed attentional resources to refine the already available strategy or strategies and/or invent new ones. Applied to our findings, it could be argued that the children were still focused on improving the procedural fluency of the DS strategy and, therefore, had no attentional resources available yet for the discovery of alternative strategies like the IA strategy.

A second explanation for the rarity of the IA strategy among the children from Study 2 and 3 focuses on their instructional histories. As argued above, the rarity of IA strategies among the children from Study 2 and those from the DS-oriented schools of Study 3 was probably due to the fact that these children had participated in an instructional practice and culture that primarily aimed at the development of routine mastery of one particular strategy, namely (the jump version of) DS. Arguably, the instructional practice and culture in these classes had not promoted (and maybe even hindered) the discovery and development of other strategies than the one preferred and taught by the textbook and the teacher. But, why did the children from the IA-oriented classes in Study 3 not exhibit efficient and flexible use of the IA strategy? Although we acknowledge the possible role of the above-mentioned general and/or specific cognitive factors, we assume that the quality of the IA-oriented instruction was still too weak to promote the development of adaptive expertise in these children. More specifically, the textbook and the teacher may have focused too much on teaching pupils how to execute the IA strategy and how to identify problems that must be solved by means of this strategy, neglecting the need of a deep understanding of the conceptual underpinnings of the IA strategy (i.e., the missing-addend relation of subtraction) and of a self-generated choice mechanism for switching adaptively between DS and IA strategies. This nicely illustrates a common problem with an instructional approach that blends elements of a skills and conceptual (rather than investigative and/or problem-solving oriented) approach to elementary mathematics education (Baroody, 2003; see also Baroody & Coslick, 1998).

Although the reviewed studies yielded new insights in the development, the characteristics, and the use of IA as a clever strategy for doing symbolically presented subtraction sums, further research is clearly needed. First, research should try to further unravel the strategy parameters of the IA strategy—in terms of frequency, efficiency, and flexibility—in people of different ages and different levels of experience in (school) arithmetic, with particular attention to young
children. By complementing assessments of the selection and use of the IA strategy with measures of their general and specific cognitive resources, we may arrive at a better understanding of why young children find it so difficult to discover and use the IA strategy as an alternative strategy for doing subtractions.

Second, the sharp contrast between the ease and success with which most young adults from Study 1 applied the IA strategy and its rarity among the children from Study 2 and 3 stresses the need for developmental studies that provide a more fine-grained picture of the process of the origin and change of the IA strategy. In this respect, the microgenetic method (Siegler, 1998) may provide a useful means for obtaining the kind of fine-grained information that is essential for such a deep understanding of the IA strategy discovery and change process.

Finally, there is a need for intervention studies in which the emergence and the further development of the IA strategy is investigated in the context of more powerful IA-oriented learning environments. Based on available research-based insights about the kind of instructional environment that is needed to promote adaptive expertise in elementary arithmetic in general (Baroody, 2003; Verschaffel, Torbeyns et al., 2007) and on the rather positive results of a few intervention studies on the IA strategy in particular (Blöte et al., 2000; Fuson & Fuson, 1992; Fuson & Willis, 1988; Klein et al., 1998), the following four design principles might be proposed: (a) integrating the development of procedural and conceptual knowledge about IA (Baroody, 2003); (b) using children’s well-documented capacity to insightfully solve context problems with a missing-addend structure (“I had 3 blocks. I got some more blocks. Now I have 8 blocks. How many blocks did I get?”) by counting or adding on from the smaller number to the larger (Fuson, 1992; De Corte & Verschaffel, 1987) as a stepping stone to the development of IA as a strategy for doing symbolically presented subtraction problems; (c) working toward a notion of strategy flexibility that is not merely conceived in terms of task features (i.e., “apply IA (only) if the difference between minuend and subtrahend is smaller than 10”) but in terms of balancing task, subject, and context features (Verschaffel, Torbeyns et al., 2007); and (d) creating a classroom culture that is supportive to the development of adequate beliefs about and attitudes towards strategy flexibility (Baroody, 2003; Verschaffel, Torbeyns et al., 2007).

REFERENCES


